



# Uniqueness in $\mathcal{E}(X, \omega)$

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## Abstract

We prove uniqueness for the Dirichlet problem for the complex Monge–Ampère equation on compact Kähler manifolds in the case of probability measures vanishing on pluripolar sets. The proof uses the mass concentration technique due to Kołodziej coupled with inequalities for mixed Monge–Ampère measures and the comparison principle.

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## 1. Introduction

Pluripotential theory on compact Kähler manifolds turned out to be a very effective tool in complex geometry and dynamics. Despite the fact that it deals with (a priori) non-smooth functions, the techniques were used with success in purely geometrical problems. For example, the  $L^\infty$  estimate from [18] gives enough flexibility for studying various limiting problems in geometry which were inaccessible with the standard (restrictive) PDE techniques.

However the ideas considered soon opened new problems of independent interest. In a series of articles Guedj, Zeriahi and collaborators [15,16,14,2,10] and Kołodziej [18,19] laid down the foundations of the theory. In particular the complex Monge–Ampère operator was defined on (non-smooth)  $\omega$ -psh functions (see the next section for definitions of all the notions appearing in the introduction), and its maximal domain of definition was explored. By analogy to the works of Błocki [4,5] and Cegrell [8,9] many results from the “flat” theory (i.e. the one in domains in  $\mathbb{C}^n$ ) were adjusted to the Kähler manifold case. However from the very beginning some prob-

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lems revealed to be unexpectedly difficult in the new setting. One of them is the problem of uniqueness.

This problem consists of the following: Let  $\phi, \psi$  be  $\omega$ -psh functions and the Monge–Ampère operator is well defined for them. Assume that  $\omega_\phi^n = \mu = \omega_\psi^n$ . The question is under what assumptions on the positive measure  $\mu$  and/or on the functions  $\phi, \psi$  one can conclude that  $\phi - \psi$  is constant?

Note that there are known examples of failure of the uniqueness in general (see [3] for one such example), hence some assumptions are necessary.

The first result in this direction was done by E. Calabi [7]. He proved that if  $\phi, \psi$  are smooth and  $\omega_\phi, \omega_\psi$  are Kähler forms (i.e. strictly positive) then uniqueness does hold. These are natural assumptions from geometer’s perspective and the proof is quite easy in this case. However both smoothness and strict positivity are crucial in this approach, hence it gives no insight what to do in general.

The next step was done by Bedford and Taylor [1] who proved uniqueness for bounded  $\phi, \psi$  provided the underlying manifold is  $\mathbb{P}^n$ . Their main idea was to control the  $L^2$  norm of the gradient of the difference of  $\phi$  and  $\psi$ .

Using different technique Kołodziej [19] proved uniqueness for bounded functions on arbitrary compact Kähler manifold modulo additional mild assumptions on the measure  $\mu$ .

The “bounded” case was finally done by Błocki [3]. The proof has some common points with the one in [1], but is much easier and transparent. Furthermore the proof gives some stability results showing that when one perturbs the measure on the right-hand side slightly the normalized solution is in a way close to the original one.

By developing theory of Cegrell classes in the Kähler manifold setting [15,16] (see also [12]) the domain of definition of the operator was enlarged with many unbounded functions. Guedj and Zeriahi [16] observed that Błocki’s argument, with suitable modifications, can be carried over to prove uniqueness in the class  $\mathcal{E}^1(X, \omega)$ . Recently Demailly and Pali [11] proved uniqueness in the same class for more general forms which are only semi-positive. The most general result so far was proven very recently by Błocki (see [6]) who proved that uniqueness does hold in the class  $\mathcal{E}^{1-\frac{1}{2^{n-1}}}(X, \omega)$ ,  $n = \dim X$ .

Simultaneously the picture in the flat theory was made much clearer by Cegrell who proved in [9] that one can prove uniqueness provided the measure  $\mu$  does not charge pluripolar sets. The proof however relied heavily on tools that are not available in the compact setting. Nevertheless it is natural to expect that uniqueness in this class (called  $\mathcal{E}(X, \omega)$ ) should also hold. In fact in [10] this point is an important obstruction for further understanding of the domain of definition of the Monge–Ampère operator.

The class  $\mathcal{E}(X, \omega)$  deserves special interest, due to the following result proven in [16]:

**Theorem 1.1.** *Let  $\mu$  be a probability Borel measure that vanishes on pluripolar sets. Then there exists (at least one)  $\rho \in \mathcal{E}(X, \omega)$  such that*

$$\omega_\rho^n = \mu, \quad \sup_X \rho = 0.$$

It is therefore important to study what happens in  $\mathcal{E}(X, \omega) \setminus \mathcal{E}^1$ , so one can understand better the action of the complex Monge–Ampère operator. Let us state our main result which solves the uniqueness problem in  $\mathcal{E}(X, \omega)$  completely:

**Theorem 1.2.** Let  $\phi, \psi \in \mathcal{E}(X, \omega)$ , be such that  $\omega_\phi^n = \omega_\psi^n$ . Then  $\phi - \psi$  is constant.

## 2. Preliminaries

Throughout the note we shall work on a fixed compact  $n$ -dimensional Kähler manifold  $X$  equipped with a fundamental Kähler form  $\omega$  (that is  $d$ -closed strictly positive globally defined form) given in local coordinates by

$$\omega = \frac{i}{2} \sum_{k,j=1}^n g_{k\bar{j}} dz^k \wedge d\bar{z}^j.$$

We assume that the metric is normalized so that

$$\int_X \omega^n = 1.$$

Recall that

$$PSH(X, \omega) := \{\phi \in L^1(X, \omega) : dd^c \phi \geq -\omega, \phi \in C^\uparrow(X)\},$$

where  $C^\uparrow(X)$  denotes the space of upper semicontinuous functions and, as usual,  $d$  is the standard operator of exterior differentiation while  $d^c := i/2\pi(\bar{\partial} - \partial)$ . We call the functions that belong to  $PSH(X, \omega)$   $\omega$ -plurisubharmonic ( $\omega$ -psh for short). We shall often use the handy notation  $\omega_\phi := \omega + dd^c \phi$ . Also, for the sake of brevity, we shall denote sets  $\{z \in X \mid u(z) > -j\}$  simply by  $\{u > -j\}$ .

The  $\omega$ -psh functions are locally standard plurisubharmonic functions minus a (smooth) potential for the form  $\omega$ . This allows to use classical local results from pluripotential theory. In particular the Monge–Ampère operator

$$\omega_\phi^n := \omega_\phi \wedge \cdots \wedge \omega_\phi$$

is well defined for bounded  $\omega$ -psh functions. This approach was used in [18,19,12].

We recall below the definition of the class  $\mathcal{E}(X, \omega)$ .

For every  $u \in PSH(X, \omega)$   $(\omega + dd^c \max(u, -j))^n$  is a well-defined probability measure. By [16] the sequence of measures  $\chi_{\{u > -j\}}(\omega + dd^c \max(u, -j))^n$  is always increasing and one defines

$$\mathcal{E}(X, \omega) := \left\{ u \in PSH(X, \omega) \mid \lim_{j \rightarrow \infty} \int_X \chi_{\{u > -j\}} (\omega + dd^c \max(u, -j))^n = 1 \right\}.$$

These functions are a priori unbounded, but the integral assumption ensures that the Monge–Ampère measure has no mass on  $\{u = -\infty\}$ . Then one defines

$$(\omega + dd^c u)^n := \lim_{j \rightarrow \infty} \chi_{\{u > -j\}} (\omega + dd^c \max(u, -j))^n.$$

In particular Monge–Ampère measures of functions from  $\mathcal{E}(X, \omega)$  do not charge pluripolar sets. We refer to [16] for a discussion of that notion.

The class  $\mathcal{E}^1(X, \omega)$ , or more generally  $\mathcal{E}^p(X, \omega)$ ,  $p > 0$ , is defined by

$$\mathcal{E}^p(X, \omega) := \left\{ \phi \in \mathcal{E}(X, \omega) \mid \int_X |\phi|^p \omega_\phi^n < \infty \right\}.$$

Since  $\omega$ -psh functions are upper semicontinuous, they are bounded from above, hence one usually considers only nonpositive  $\omega$ -psh functions from  $\mathcal{E}^p(X, \omega)$ , which often comes in handy in technical details. Note that originally the classes  $\mathcal{E}^p$  were defined (similarly to the Cegrell classes in the flat theory) with the use of a sequence of bounded functions  $\phi_j$ ,  $\phi_j \searrow \phi$ , such that  $\sup_j \int_X |\phi_j|^p \omega_{\phi_j}^n < \infty$ . The results from [16,12] have shown that actually one can take just the sequence  $\phi_j := \max(\phi, -j)$ , hence both definitions are coherent.

One can also define “local” classes in an attempt similar to the one from [4,5]. We define the class  $\mathcal{D}(X, \omega)$  by

$$\mathcal{D}(X, \omega) := \{ \phi \in \text{PSH}(X, \omega) \mid \forall z \in X \exists U_z \text{-open, } z \in U_z, \rho + \phi \in \mathcal{D}(U_z) \},$$

where  $\rho$  is a local potential in  $U_z$  for  $\omega$  and  $\mathcal{D}(U_z)$  is the maximal domain of definition of the Monge–Ampère operator in  $U_z$  (see [4,5]).

Note however that the “local” and global definition yield different classes, as shown in [16] (this was also studied in [10]). This is in sharp contrast with the “flat” theory.

Define also  $\mathcal{D}^a(X, \omega)$  by

$$\mathcal{D}^a(X, \omega) := \{ \phi \in \mathcal{D}(X, \omega) \mid \omega_\phi^n(A) = 0, \forall A \subset X, A \text{-pluripolar} \}.$$

It is known that  $\mathcal{D}^a(X, \omega) \subset \mathcal{E}(X, \omega)$ , while  $\mathcal{D}(X, \omega) \not\subset \mathcal{E}(X, \omega) \not\subset \mathcal{D}(X, \omega)$ .

Please note that the terminology in the Cegrell classes, partially due to the mentioned differences in the “local” and “global” settings varies in the literature. In particular the class  $\mathcal{D}(X, \omega)$  is denoted by  $\mathcal{E}(X, \omega)$  in [17] or [13]. The class  $\mathcal{E}(X, \omega)$  in turn differs in some aspects from the class  $\mathcal{E}(\Omega)$  in the “flat” setting (for example a function in  $\mathcal{E}(\Omega)$  may have a Monge–Ampère measure that charges points).

The first result that we shall need – an inequality for mixed Monge–Ampère measures – was shown in a special case in [19] and in full generality in [13].

**Theorem 2.1.** *Let  $u, v \in \mathcal{E}(X, \omega)$  be  $\omega$ -psh functions,  $\mu$  be a positive measure that does not charge pluripolar sets and  $f, g \in L^1(d\mu)$ . If*

$$(\omega + dd^c u)^n \geq f d\mu, \quad (\omega + dd^c v)^n \geq g d\mu$$

*as measures, then*

$$(\omega + dd^c u)^k \wedge (\omega + dd^c v)^{n-k} \geq f^{\frac{k}{n}} g^{\frac{n-k}{n}} d\mu, \quad \forall k \in \{1, \dots, n-1\}.$$

**Corollary 2.2.** *If  $\phi, \psi \in \mathcal{E}(X, \omega)$  and  $\omega_\phi^n = \omega_\psi^n$ , then also for every  $t \in (0, 1)$  we have*

$$\omega_{t\phi+(1-t)\psi}^n = \omega_\phi^n = \omega_\psi^n.$$

The second result we shall need is somewhat nonstandard, so although it is similar to the usual comparison principle, we give a detailed proof.

**Theorem 2.3** (“Partial” comparison principle). *Suppose  $T$  is a  $(k, k)$  positive closed current on  $X$  of the form  $\omega_{\phi_1} \wedge \cdots \wedge \omega_{\phi_k}$ ,  $\phi_j \in \mathcal{E}(X, \omega) \forall j \in \{1, \dots, k\}$ , where  $0 \leq k \leq n-1$ . Let furthermore  $u, v \in \mathcal{E}(X, \omega)$ . Then*

$$\int_{\{u < v\}} \omega_v^{n-k} \wedge T \leq \int_{\{u < v\}} \omega_u^{n-k} \wedge T.$$

**Proof.** Note that in the case  $k = 0$  this is the standard comparison principle in  $\mathcal{E}(X, \omega)$ , which was shown in Theorem 1.5 in [16] (historically the bounded case was first shown in [19]). Note also that it is enough to get the statement for  $n - k = 1$ , since all the other cases can be done by iteration of the partial comparison principle for  $n - k = 1$ , while the other factors of type  $\omega_v^s \wedge \omega^{n-k-s-1}$  are regarded as part of  $T$ . So, in the sequel we assume  $n - k = 1$ .

In the class  $\mathcal{D}^a(X, \omega)$  the partial comparison principle (for  $n - k = 1$ ) was proven in [17]. In particular the result holds for bounded  $u, v$  and  $\phi_j, j = 1, \dots, k$ .

Suppose now that  $u$  and  $v$  are bounded. Define the canonical approximants for  $\phi_1, \dots, \phi_{n-1}$  by

$$\phi_s^{(j)} := \max(\phi_s, -j).$$

Let  $T^{(j)} := \omega_{\phi_1^{(j)}} \wedge \cdots \wedge \omega_{\phi_{n-1}^{(j)}}$ . Note that all the functions which appear in the wedge products belong to  $\mathcal{D}^a(X, \omega)$ , hence [17] applies. So, for  $u, v$  bounded we have

$$\int_{\{u < v\}} \omega_v \wedge T^{(j)} \leq \int_{\{u < v\}} \omega_u \wedge T^{(j)}.$$

Now letting  $j \rightarrow +\infty$  (and using continuity properties of the Monge–Ampère operator defined in the class  $\mathcal{E}(X, \omega)$  proven in [16]) one gets

$$\int_{\{u < v\}} \omega_v \wedge T \leq \int_{\{u < v\}} \omega_u \wedge T.$$

Finally we relax the boundedness condition on  $u, v$  exactly as in Theorem 1.5 in [16]. We include the argument for the sake of completeness. Namely we set  $u_j = \max(u, -j)$  and  $v_j = \max(v, -j)$ . It follows that for any  $k, j > 0$

$$\int_{\{u_j < v_k\}} \omega_{v_k} \wedge T \leq \int_{\{u_j < v_k\}} \omega_{u_j} \wedge T.$$

Using the inclusions  $\{u_j < v\} \subset \{u_j < v_k\} \subset \{u < v_k\}$  and letting  $k \rightarrow +\infty$  one obtains

$$\int_{\{u_j < v\}} \omega_v \wedge T \leq \int_{\{u \leq v\}} \omega_{u_j} \wedge T.$$

Now, in turn, we let  $j \rightarrow +\infty$  and obtain  $\int_{\{u < v\}} \omega_v \wedge T \leq \int_{\{u \leq v\}} \omega_u \wedge T$ . If we replace  $v$  by  $v - \epsilon$ ,  $\epsilon > 0$  we get  $\int_{\{u < v - \epsilon\}} \omega_v \wedge T \leq \int_{\{u \leq v - \epsilon\}} \omega_u \wedge T$ . Finally letting  $\epsilon \rightarrow 0^+$  we get the claimed result.  $\square$

**Remark.** It is likely that the result above holds for every positive closed current  $T$  of bidegree  $(k, k)$ . Also it is likely that the partial comparison principle should hold for more general functions than the  $\omega$ -psh ones and this new class should depend on  $T$ . We shall not go into details, since the stated version is satisfactory for our needs.

### 3. Proof of the main result

**Proof.** Consider the level sets

$$A_t := \{\phi - \psi = t\}, \quad t \in \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}.$$

These are all Borel sets which are closed in the plurifine topology. The main ingredient of the proof is to show that the whole mass of  $\mu := \omega_\phi^n = \omega_\psi^n$  is concentrated on exactly **one** of the sets  $A_t$ . To achieve this we need some additional results.

**Lemma 3.1.** *The measure charges at most countably many of the sets  $A_t$  and does not charge neither  $A_{+\infty}$  nor  $A_{-\infty}$ .*

**Proof.** The first part is proved in [16, Corollary 1.10], and the second follow from the fact that both  $A_{+\infty}$  and  $A_{-\infty}$  are pluripolar.  $\square$

Suppose now that the mass of  $\mu$  is not concentrated on one set  $A_t$ . Then we claim that we can find  $t_0 \in \mathbb{R}$  and a constant  $1/2 < q < 1$  such that:

- (1)  $\int_{A_{t_0}} d\mu = 0$ ,
- (2)  $\int_{\{\phi < \psi + t_0\}} d\mu < q$ ,
- (3)  $\int_{\{\phi > \psi + t_0\}} d\mu < q$ .

Indeed, one can find  $t_1 \in \mathbb{R}$  such that

$$0 < \int_{\{\phi < \psi + t_1\}} d\mu < 1$$

(for otherwise the whole mass is concentrated on one level set). Now if  $\int_{A_{t_1}} d\mu = 0$ , we take  $t_0 := t_1$ ,  $q = \max\{\int_{\{\phi < \psi + t_1\}} d\mu, 1 - \int_{\{\phi < \psi + t_1\}} d\mu\} + \epsilon$ , with  $\epsilon > 0$  so small that still  $q < 1$ . If  $A_{t_1}$  is charged, then for almost every  $t < t_1$ ,  $A_t$  is not charged and by monotone convergence one can take  $t_2 < t_1$  close enough to  $t_1$  such that both  $A_{t_2}$  is massless and still  $0 < \int_{\{\phi < \psi + t_1\}} d\mu < 1$ . Take  $t_0 := t_2$ ,  $q$  defined as before and we get the desired properties.

Since adding a constant to  $\phi$  or  $\psi$  is harmless for our discussion we assume from now on that  $t_0 = 0$ .

Consider the new measure

$$\widehat{\mu} := \begin{cases} (1/q)\mu & \text{on } \{\phi < \psi\}, \\ c\mu & \text{on } \{\phi \geq \psi\}, \end{cases}$$

where  $c$  is a nonnegative normalization constant so that  $\widehat{\mu}$  is a nonnegative probability measure (note that this is possible, since, by assumption,  $\mu$  charges the set  $\{\phi \geq \psi\}$ ).

Of course  $\widehat{\mu}$  does not charge pluripolar sets either (and is also a Borel measure since the set  $\{\phi \geq \psi\}$  is Borel). By [16] we can solve the Monge–Ampère equation

$$\omega_\rho^n = \widehat{\mu}, \quad \rho \in \mathcal{E}(X, \omega), \quad \sup_X \rho = 0.$$

Note that at this moment we do not know if  $\rho$  is uniquely defined: we just choose one solution.

In such a case we have a set inclusion

$$U_t := \{(1-t)\phi < (1-t)\psi + t\rho\} \subset \{\phi < \psi\}$$

for every  $t \in (0, 1)$ . Hence on  $U_t$  we have

$$\omega_\phi^{n-1} \wedge \omega_{(1-t)\psi + t\rho} = (1-t)\mu + t\omega_\phi^{n-1} \wedge \omega_\rho \geq (1 + ((1/q)^{1/n} - 1)t)\omega_\phi^n,$$

where we have made use of Theorem 2.1 and Corollary 2.2.

So by the comparison principle

$$\begin{aligned} (1 + ((1/q)^{1/n} - 1)t) \int_{U_t} \omega_\phi^n &\leq \int_{U_t} \omega_\phi^{n-1} \wedge \omega_{(1-t)\psi + t\rho} \\ &\leq \int_{U_t} \omega_\phi^{n-1} \wedge \omega_{(1-t)\phi + t0} = (1-t) \int_{U_t} \omega_\phi^n + t \int_{U_t} \omega_\phi^{n-1} \wedge \omega. \end{aligned}$$

Rearranging terms we obtain

$$(1/q)^{1/n} \int_{U_t} \omega_\phi^n \leq \int_{U_t} \omega_\phi^{n-1} \wedge \omega. \quad (3.1)$$

Note that exchanging  $\omega_\phi^{n-1}$  with  $\omega_\psi^{n-1}$  in the argument above gives

$$(1/q)^{1/n} \int_{U_t} \omega_\psi^n \leq \int_{U_t} \omega_\psi^{n-1} \wedge \omega \quad (3.2)$$

(again we make use of Corollary 2.2 here). Now let  $t \searrow 0$ . The sets  $U_t$  form an increasing sequence, and  $U_t \nearrow \{\phi < \psi\} \setminus \{\rho = -\infty\}$ . But both measures  $\omega_\phi^n$  and  $\omega_\phi^{n-1} \wedge \omega$  do not charge pluripolar sets, hence we obtain

$$(1/q)^{1/n} \int_{\{\phi < \psi\}} \omega_\phi^n \leq \int_{\{\phi < \psi\}} \omega_\phi^{n-1} \wedge \omega. \quad (3.3)$$

One can do this reasoning also on the set  $\{\phi > \psi\}$ . Namely we find a measure defined like  $\widehat{\mu}$ , but with respect to the set  $\{\phi > \psi\}$ . Fixing  $\omega_\phi^{n-1}$  (or  $\omega_\psi^{n-1}$ ) and arguing the same way we obtain

$$(1/q)^{1/n} \int_{\{\phi > \psi\}} \omega_\phi^n \leq \int_{\{\phi > \psi\}} \omega_\phi^{n-1} \wedge \omega. \quad (3.4)$$

But adding these inequalities and the assumption that  $A_0$  is massless one obtains

$$\begin{aligned} (1/q)^{1/n} &= (1/q)^{1/n} \int_{\{\phi > \psi\}} \omega_\phi^n + (1/q)^{1/n} \int_{\{\phi < \psi\}} \omega_\phi^n \\ &\leq \int_{\{\phi > \psi\}} \omega_\phi^{n-1} \wedge \omega + \int_{\{\phi < \psi\}} \omega_\phi^{n-1} \wedge \omega \leq 1, \end{aligned}$$

a contradiction.

So we can assume that the whole mass of  $\mu$  is concentrated on  $\{\phi = \psi\} \neq X$ .

The next step will be to prove that the same holds for all the measures

$$\omega_\phi^k \wedge \omega_\psi^{n-1-k} \wedge \omega, \quad k \in \{0, \dots, n-1\}.$$

Note that for these measures one cannot apply straightforwardly the inequality for mixed Monge–Ampère measures, hence we do not have that these measures are (a priori) globally equal. But we shall need only the fact that they all are massless on  $\{\phi \neq \psi\}$  – and this is exactly what the argument will give.

Let  $\phi_j := \max\{\phi, -j\}$ . Fix  $t \in (0, 1)$ . Consider the sets

$$V_{t,j} := \{\phi + (t/j)\phi_j + (3/2)t < \psi\} \subset \{\phi < \psi\}.$$

Again by the comparison principle we obtain

$$\int_{V_{t,j}} \omega_\phi^k \wedge \omega_\psi^{n-1-k} \wedge (\omega_\psi + (t/j)\omega) \leq \int_{V_{t,j}} \omega_\phi^k \wedge \omega_\psi^{n-1-k} \wedge (\omega_\phi + (t/j)\omega_{\phi_j}).$$

Now (recall  $\omega_\phi^k \wedge \omega_\psi^{n-k} = \omega_\phi^n$ ,  $\forall k \in \{0, \dots, n-1\}$ ), the equation above reads

$$\int_{V_{t,j}} \omega_\phi^k \wedge \omega_\psi^{n-1-k} \wedge \omega \leq \int_{V_{t,j}} \omega_\phi^k \wedge \omega_\psi^{n-1-k} \wedge \omega_{\phi_j}.$$

Note that  $V_{t,j}$  is a decreasing sequence of sets in terms of  $j$ . Letting  $j \rightarrow \infty$  and using vanishing on pluripolar sets we obtain

$$\int_{\{\phi + (3/2)t < \psi\}} \omega_\phi^k \wedge \omega_\psi^{n-1-k} \wedge \omega \leq \int_{\{\phi + (3/2)t < \psi\}} \omega_\phi^n = 0.$$



Finally letting  $t \searrow 0$  we obtain

$$\int_{\{\phi < \psi\}} \omega_{\phi}^k \wedge \omega_{\psi}^{n-1-k} \wedge \omega \leq \int_{\{\phi < \psi\}} \omega_{\phi}^n = 0.$$

Again exchanging  $\phi$  with  $\psi$  and  $\{\phi < \psi\}$  with  $\{\phi > \psi\}$  one obtains that the measures  $\omega_{\phi}^k \wedge \omega_{\psi}^{n-1-k} \wedge \omega$  are massless on  $\{\phi \neq \psi\}$ .

But then exchanging  $\omega_{\phi}^k \wedge \omega_{\psi}^{n-1-k}$  in the argument above with  $\omega_{\phi}^k \wedge \omega_{\psi}^{n-2-k} \wedge \omega$  we obtain that  $\omega_{\phi}^k \wedge \omega_{\psi}^{n-2-k} \wedge \omega^2$  is again massless on  $\{\phi \neq \psi\}$  (this time we do not have equality for the mixed Monge–Ampère measures in the whole  $X$  but instead we use vanishing on  $\{\phi \neq \psi\}$ ).

Iterating the argument yields that all the measures

$$\omega_{\phi}^k \wedge \omega_{\psi}^s \wedge \omega^{n-k-s}$$

are massless on  $\{\phi \neq \psi\}$ . Hence this holds also for  $\omega^n$  i.e.

$$\int_{\{\phi \neq \psi\}} \omega^n = 0.$$

But  $\{\phi \neq \psi\}$  is a plurifine open set, hence (if it is non-empty) it has positive Lebesgue measure. This is a contradiction.  $\square$

#### 4. Observations and corollary

**Observation 4.1.** *In the proof of uniqueness we have not made use of the full strength of the Kähler condition on  $\omega$ . Note that if  $\omega$  is merely **big** i.e.  $\omega$  is a smooth nonnegative definite  $(1, 1)$ -form satisfying  $\int_X \omega^n > 0$ , the same argument will go through, provided the set  $\omega^n = 0$  is of zero Lebesgue measure (for a discussion of the case of semi-positive forms we refer to [2,14,11]). The proof carries through even if we allow merely continuous (instead of smooth) potentials for  $\omega$  (in this situation nonnegativity is understood in the weak sense), but the final argument cannot be done without any assumption on the measure of the degeneration set.*

**Observation 4.2.** *The assumptions from the observation above are exactly the same as those from [11]. So the result is also a generalization of the Demailly–Pali uniqueness in the big case. Note also that the assumptions are satisfied in many cases that are of interest for dynamics and geometry: for example if  $\omega$  is a pull-back of a Kähler form via proper holomorphic mapping, the degeneration set is analytic, hence of zero Lebesgue measure.*

Finally we would like to point out the following corollary, which was already observed by Blocki in [6]:

**Corollary 4.3.** *We have that  $\mathcal{D}^a(X, \omega) \subsetneq \mathcal{E}(X, \omega)$ , so it follows that given a probability measure  $\mu$  on  $X$  the Dirichlet problem*

$$\omega_{\phi}^n = \mu, \quad \sup_X \phi = 0, \quad \phi \in \mathcal{D}^a(X, \omega)$$

*need not have a solution in general. This is in sharp contrast with the Dirichlet problem in the bounded case, since it is proven in [20], that being a Monge–Ampère measure of a bounded  $\omega$ -psh function is a local property.*

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